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Hölder continuity of solutions to elastic traffic network models

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Abstract This paper aims to study stability and sensitivity analysis for quasi-variational inequalities which model traffic network equilibrium problems with elastic travel demand. In particular, we provide a Hölder stability result under parametric perturbations.

Keywords Quasi-variational inequalities · Hölder continuity · Traffic equilibrium problem

1 Introduction

Our aim is to deal with stability and sensitivity analysis for a class of quasi-variational inequalities modeling elastic traffic equilibrium problems. We focus our attention on the continuity of solutions subject to parametric dependence, and obtain a Hölder stability result, which can be viewed as a first step for differentiability analysis. Although we refer to elastic traffic models (see [7,8,16] for a discussion on such models), the formulation of the quasi-variational inequality problem we consider may encompass numerous applied problems, such as generalized Nash equilibria, multi-leader-follower games, superconductivity, thermoplasticity, and electrostatic with implicit ionization threshold.

Continuity of solutions has a central position; in fact it arises in different fields, such as in spacial market equilibrium, Nash equilibria, oligopolistic equilibrium models, traffic equilibrium problems, and optimal control. Common practical applications include energy planing, urban transit system analysis and design, and prediction of intercity freight flows.

Sensitivity analysis was pioneered in the context of nonlinear programming in the seminal work by Fiacco and McCormick [9] and later enlarged in [10,11,17]. In a parallel fashion,

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sensitivity analysis was applied for variational inequalities, reaching a very high level, especially after the works of Robinson [14,15] on generalized equations, which have allowed a new and unified perspective for both variational inequalities and nonlinear programming.

Thus, prompted by the large number of applications, numerous mathematicians investigated many aspects of sensitivity and stability analysis for variational inequalities, optimization problems and inequality systems. For a survey of this theory, we refer, among the others, to [1–3,12,13,18,22].

However, to the best of our knowledge, the continuity of the solution map for quasi-variational inequalities has not been thoroughly studied, and only few efforts have been devoted to this topic. In particular, the Hölder or Lipschitz continuity for this class of problems is still missing as confirmed by the results in the literature. Moreover, the stability issue in transportation problems still remains a challenging question. User equilibrium always depends on some parameters, the most common of which are monetary cost (including tolls and fuel consumption), distance, rate of accidents, and general travel comfort. These circumstances naturally influence the preference options of traffic users, who then need a monitoring of their decision-making process. In many situations, some of the factors involved in transportation networks may be regarded as perturbing parameters rather than real criteria. For this reasons, we were motivated to cope with Hölder stability issues in the framework of traffic equilibrium problems expressed in terms of quasi-variational inequalities.

We now outline the remainder of the paper. Section 2 is devoted to the detailed description of the traffic network model. In Sect. 3, a Hölder continuity property of data as functions of the parameters is given, and, finally, in Sect. 4 the main result is discussed.

2 The traffic equilibrium model

The foundation of the study of traffic network problems goes back to Wardrop [21], who stated the basic equilibrium principle in 1952. Over the past decades, a large number of efforts have been devoted to the study of traffic assignment models, with emphasis on efficiency, and optimality, in order to improve practicability, reduce gas emissions, and contribute to the welfare of the community. The variational inequality approach to such problems begins with the seminal work of Smith [19], who proved that the user-optimized equilibrium can be expressed in terms of a variational inequality. Thus, the possibility of exploiting the powerful tools of variational analysis has led to deal with a large variety of models, reaching valuable theoretical results and providing applications in practical situations. In this paper, we are concerned with a class of equilibrium problems which can be studied in the framework of quasi-variational inequalities, see [7,8]. In particular, we focus on models with elastic travel demands, in the sense that they depend on the equilibrium distribution.

Let us consider the graph of a network $G = [N, L]$, where N is the set of nodes (e.g., crossroads, airports, railway stations) and L is the set of directed links (stretches of streets). Let a denote a link of the network connecting a pair of nodes and let p be a path consisting of a sequence of links which connect an Origin–Destination (O/D) pair of nodes. We suppose that in the network there are n links and m paths. Let W be the set of O/D pairs with typical O/D pair w; we also suppose that the number of elements of W is l, with $m > l$. The set of paths connecting the O/D pair w is denoted by P_w and the set of all the paths in the network by $\mathcal{P}$. Let f_a denote the traffic flow on link a and v_p the non-negative traffic flow on path p. Let $f = (f_1, \dots, f_n)^T$ be the link flow vector and $v = (v_1, \dots, v_m)^T$ the path flow vector. The relationship between the links and the path flows is given by $f_a = \sum_{p \in \mathcal{P}} \delta_{ap} v_p$ or $f = \Delta v$, where $\Delta = (\delta_{ap})_{a \in L, p \in \mathcal{P}}$ is the link-path incidence matrix, whose typical entry

δ_{ap} is 1 if link a is contained in path p and 0 otherwise. The above condition shows that the flow on a particular link is given by the sum of the flows on the paths which contain that link. We assume that the travel demand associated with the users traveling between O/D pairs is not fixed, but depends on the equilibrium distribution denoted by u. In fact, it is clear that travel demands are affected by users' prediction of the flow amount through the network. Let E be a nonempty, compact, and convex subset of $\mathbb{R}^m_+$ and let $d\colon E \to \mathbb{R}^l_+$ the travel demand. The set of feasible flows is the set-valued map $K\colon E \to 2^{\mathbb{R}^m_+}$:

$$K(u) = \{v \in E : Av = d(u)\},$$

where $A = (\phi_{wp})_{w\in W, p\in P}$ is the link-route incidence matrix O/D pairs-paths whose typical entry ϕ_{wp} is 1 if path p connects the pair w and 0 otherwise. The meaning of the conservation condition $Av = d(u)$ is that flows and hence travelers are not lost or generated in the network. Let $c_a(f)$ denote the user travel cost associated with the link a and group the link costs into the vector $c(f) = (c_1(f), \ldots, c_n(f))^T$. Only the case of asymmetric cost, i.e., the cost on a link does not depend only on the flow on that link, but it is affected by the flows on all the links in the network, will be taken into consideration. Let $C_p(v)$ denote the user travel cost on path p and let $C(v) = (C_1(v), \ldots, C_m(v))^T$ be the path flow vector. It results: $C_p(v) = \sum_{a\in L} \delta_{ap} c_a(f)$ or $C(v) = \Delta^T c(f) = \Delta^T c(\Delta v)$. The above relationship shows that the cost on a path is given by the sum of the costs on links which form the path.

Now we provide the definition of equilibrium distribution, extending the well-known user equilibrium flow as introduced in Wardrop [21], and its equivalent quasi-variational inequality formulation.

Definition 1 A flow $u \in K(u)$ is a user traffic equilibrium flow if $\forall w \in W$ and $\forall p, s \in P_w$ it results that

$$C_p(u) > C_s(u) \Longrightarrow u_p = 0.$$

Theorem 1 *A flow $u \in K(u)$ is an equilibrium pattern if and only if it satisfies the following quasi-variational inequality*

$$\langle C(u), v - u\rangle \geq 0, \quad \forall v \in K(u). \tag{1}$$

Remark 1 It is reasonable to assume that $d(u) > 0$, $\forall u \in E$, because, otherwise, the network would be empty. In addition, there exists $\delta \in \mathbb{R}^l_+$ such that $d(u) \leq \delta$, $\forall u \in E$. In fact, for the clear physical meaning, an unlimited demand requirement would be unrealistic. Thus, it results that flows are nontrivial and norm-bounded from below. The typical set of feasible flows with h O/D pairs is as follows

$$K_d = \Big\{v \in \mathbb{R}^m : v_j \geq 0, j = 1, \ldots, m : \sum_{j\in J_1} v_j = d_1(u), \ldots, \sum_{j\in J_h} v_j = d_h(u)\Big\},$$

where J_i, $i = 1, \ldots, h$, is the set of paths $j \in \{1, \ldots, m\}$ which connect O/D pairs w_i, $i = 1, \ldots, h$. Now we compute $\min_{v\in K_d} \|v\|$. Given the Lagrangian function

$$\mathcal{L}(v, \varphi, \psi) = \sum_{j=1}^m v_j^2 - \sum_{j=1}^m \varphi_j v_j - \psi_1\Big(\sum_{j\in J_1} v_j - d_1(u)\Big) - \cdots - \psi_h\Big(\sum_{j\in J_h} v_j - d_h(u)\Big)$$

the following conditions hold

$$\begin{aligned}
&2v_j - \varphi_j - \psi_i = 0,\ j \in J_i, i = 1\ldots, h,\\
&\varphi_j v_j = 0, \varphi_j \geq 0, \quad j = 1\ldots, m,\\
&\textstyle\sum_{j\in J_i} v_j = d_i(u), \quad i = 1\ldots, h.
\end{aligned}$$

For a fixed $j \in J^i$, $i = 1 \ldots, h$, it follows that if $v_j > 0$, then $\varphi_j = 0$ and $v_j = \psi_i/2 > 0$. On the other hand, if $v_j = 0$, then $\varphi_j \geq 0$ and $\varphi_j + \psi_i/2 = 0$, which is an absurd assertion. Thus, v_j is always strictly positive. Solving the above system, we find the solution $v_j^* = \frac{d_i(u)}{s_i}$, $j \in J_i$, $i = 1, \ldots, h$, where s_i is the number of elements of J_i. Finally, after some calculations, we find that $\min_{v \in K_d} \|v\|^2 = \|v^*\|^2 = \sum_{i=1}^{h} \frac{d_i^2(u)}{s_i} \leq \sum_{i=1}^{h} \frac{\delta_i^2}{s_i}$. In the following $\sum_{i=1}^{h} \frac{\delta_i^2}{s_i}$ will be denoted by δ_0.

3 The mathematical model and related assumptions

The quasi-variational inequality subject to our treatment consists in seeking $u \in K(u)$ such that

$$\text{(QVI)} \quad \langle C(u), v - u \rangle \geq 0, \quad \forall v \in K(u),$$

where $K(u) = \{v \in E : Av = d(u)\}$, E is a convex and compact subset of $\mathbb{R}_+^m$, A is an appropriate $l \times m-$matrix (with $m > l$), and d: $E \to \mathbb{R}_+^l$ the travel demand map. Since in our model flows are nontrivial, we assume that $0 \notin E$. Therefore, as the norm is lower semicontinuous on E, feasible flows are bounded from below in norm, notice that in Remark 1 we computed the greatest lower bound for flows. We shall also assume that the solutions set to (QVI) is nonempty and nontrivial. Thus, we denote by $\bar{u}$ a solution to (QVI) belonging to some neighborhood $X \subset E$. In order to state the parametric quasi-variational inequality, we assume that C is subject to change, which can be seen in a general perturbation form by involving a parameter μ, where μ belongs to a subset of a finite dimensional space Λ, whose norm is denoted by $\|\cdot\|$. Therefore, we consider a family of cost operators $C(\cdot, \mu)_\mu$ defined from E into $\mathbb{R}_+^m$. The perturbation of constraints will be done with respect to the map d, whereas matrix A will be fixed. Hence, we suppose that there exists a parameter λ, element of a subset M of an Euclidian subspace whose norm is also denoted by $\|\cdot\|$, which acts on d. For any parameters μ and λ, $\mathcal{V}(\mu)$ and $\mathcal{V}(\lambda)$ will denote a neighborhood of μ and λ, respectively. Moreover, the initial values of λ and μ are denoted by $\bar{\lambda}$ and $\bar{\mu}$, respectively.

Therefore, for $\mu \in \mathcal{V}(\bar{\mu})$ and $\lambda \in \mathcal{V}(\bar{\lambda})$, the perturbed problem can be stated as follows: Find $u(\mu, \lambda) \in K_\lambda(u(\mu, \lambda))$ such that

$$(\text{QVI}_{\mu,\lambda}) \quad \langle C(u(\mu, \lambda), \mu), v - u(\mu, \lambda) \rangle \geq 0, \quad \forall v \in K_\lambda(u(\mu, \lambda)).$$

The above problem can be considered as a perturbed form of the problem (QVI), so that $\bar{u} = u(\bar{\mu}, \bar{\lambda})$ is a solution to $(\text{QVI}_{\bar{\mu},\bar{\lambda}})$. In the following, we assume that $(\text{QVI}_{\mu,\lambda})$ admits at least a solution $u(\mu, \lambda)$ in $K(\lambda) \cap X$. We are not interested in discussing existence issues, however, we address the interested reader to [4–6] for a discussion on this topic.

Our analysis will be carried out on the basis of the following assumptions:

(h_0) d is Hölder continuous, i.e., for some $L_1, L_2 > 0$ and $\xi, \xi' \in]0, 1[$,

$$|d_\lambda(u) - d_{\lambda'}(v)| \leq L_1 \|\lambda - \lambda'\|^{\xi'} + L_2 |u - v|^{\xi}, \ \forall u, v \in X, \ \forall \lambda, \lambda' \in \mathcal{V}(\bar{\lambda});$$

(h_1) C is uniformly strongly monotone, i.e., for some $m > 0$,

$$\langle C(u, \mu) - C(v, \mu), u - v \rangle \geq m |u - v|^2, \ \forall u, v \in X, \ \forall \mu \in \mathcal{V}(\bar{\mu});$$

(h_2) for some $b_0 > 0$, for all $\mu \in \mathcal{V}(\bar{\mu})$ and all $u \in X$ one has $|C(u, \mu)| \leq b_0$;

(h_3) for some $\gamma \in]0, 1[$ and $c > 0$, $\mu \mapsto C(\cdot, \mu)$ is uniformly (in u)(γ, c)-Hölder, i.e., for all $u \in X$ and all $\mu, \mu' \in \mathcal{V}(\bar{\mu})$,

$$|C(u, \mu) - C(u, \mu')| \leq c\|\mu - \mu'\|^{\gamma}.$$

3.1 Analysis of constraints

Before going on the analysis of parametric solutions to our problem (QVI), a crucial point is to determine how the constraints behave whenever the parameter acting on d and the point u are moving. In Proposition 1, we show that, for a Hölder continuous map d, we can dispose of a Hölder-type behavior of the map $(u, d) \mapsto K_d(u)$, which is a more strengthened property than the well-known Aubin Lipschitz property (see Aubin [3]). First, we will use the result in [20], which can be seen as a particular case of famous Hoffman's Lemma, to derive the following

Lemma 1 *Let A be an $l \times m-$ matrix, κ_1 and κ_2 be given vectors in $\mathbb{R}^l$. The solution set of the linear equality $Ax = \kappa_i$, for $i = 1, 2$, is denoted by S_i. Then, there exists $\vartheta = \vartheta(A) > 0$ such that for each $x_1 \in S_1$ there exists $x_2 \in S_2$ satisfying $|x_1 - x_2| \leq \vartheta|\kappa_1 - \kappa_2|$.*

Proposition 1 *Assume that (h_0) holds. Then, there exist $k_1, k_2 > 0$ such that $\forall \lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$, and $\forall u, v \in E$ one has:*

$$K_{\lambda}(u) \subset K_{\lambda'}(v) + (k_1\|\lambda - \lambda'\|^{\xi'} + k_2|u - v|^{\xi})\bar{\mathbb{B}}_m, \tag{2}$$

where $\bar{\mathbb{B}}_m$ denotes the unit closed ball in $\mathbb{R}^m$.

Proof Let $\lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$ and $u, v \in E$. Consider the systems: $Az = d_{\lambda}(u)$; $Az = d_{\lambda'}(u)$. From Lemma 1, there exists $\vartheta = \vartheta(A)$ such that each $z \in K_{\lambda}(u)$, we can find $z' \in K_{\lambda'}(v)$ satisfying $|z - z'| \leq \theta|d_{\lambda}(u) - d_{\lambda'}(v)|$. We now involve (h_0) and see that (2) is verified with $k_i = \theta L_i$ for $i = 1, 2$, completing the proof. □

Remark 2 In the proof of our main result, we need only the following weak requirement on the constraints which can be deduce from (2):

$$K_{\lambda}(u) \cap X \subset K_{\lambda'}(v) + (k_1\|\lambda - \lambda'\|^{\xi'} + k_2|u - v|^{\xi})\bar{\mathbb{B}}_m, \quad \forall u, v \in X. \tag{3}$$

Lemma 2 *Assume that (h_0) hold. Then, for all $u, v \in X$, for $\lambda \in \mathcal{V}(\bar{\lambda})$ and for all $\theta \in K_{\lambda}(u) \cap X$, there exists $\pi \in K_{\lambda}(v)$ such that:*

(i) $|\theta - \pi| \leq k_2|u - v|^{\xi}$;
(ii) *either* $\pi = \theta$ *or* $|\pi - \theta| \geq \frac{\delta_0}{2}$.

$\delta_0 > 0$ being the bound of flow minimal norm computed in Remark 1.

Proof The proof can be straightforwardly derived from Proposition 1 and is left to the reader. □

Remark 3 Observe that, for any $\beta \geq 2$, Remark 1 and assumption (h_2) imply that

$$\left|\langle C(u, \mu), v\rangle\right| \leq b|v|^{\beta}, \quad \forall u, v \in X : |v| \geq \frac{\delta_0}{2}, \quad \forall \mu \in \mathcal{V}(\bar{\mu}). \tag{4}$$

Here, $b = (\frac{2}{\delta_0})^{\beta-1}b_0$. Notice that (4) is trivially verified with $\beta \geq 1$.

4 Main result

Now, we are able to state and prove our main result.

Theorem 2 *Assume that* $\bar{u} = u(\bar{\mu}, \bar{\lambda})$ *is a solution to* $(QVI) = (QVI_{\bar{\mu},\bar{\lambda}})$*, conditions* (h_0)*–*(h_4) *hold and* $m > 2bk_2^{\beta}$*, where* $b = (\frac{2}{\delta_0})^{\beta-1}b_0$ *and* $\beta = \frac{2}{\xi}$*. Then, the solution* $u(\mu, \lambda)$ *to* $(QVI_{\mu,\lambda})$ *is unique in X and verifies the following condition: there exist* $c_1, c_2 > 0$, $d_1, d_2 \in]0, 1[$ *such that*

$$|u(\mu, \lambda) - u(\mu', \lambda')| \leq c_1\|\mu - \mu'\|^{d_1} + c_2\|\lambda - \lambda'\|^{d_2} \tag{5}$$

for all $\mu, \mu' \in \mathcal{V}(\bar{\mu})$, $\lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$.

Proof The proof is organized in three steps. Let us fix $\lambda \in \mathcal{V}(\bar{\lambda})$. For the two solutions $u(\mu, \lambda)$ and $u(\mu', \lambda)$, we provide an estimation of $|u(\mu, \lambda) - u(\mu', \lambda)|$ for μ and μ' around $\bar{\mu}$. Since $u(\mu, \lambda)$ is a solution to $\mathrm{QVI}_{\mu,\lambda}$, for all $v \in K_\lambda(u(\mu, \lambda))$ we have:

$$\langle C(u(\mu, \lambda), \mu), v - u(\mu, \lambda)\rangle \geq 0. \tag{6}$$

In a similar way we have for all $v \in K_\lambda(u(\mu', \lambda))$,

$$\langle C(u(\mu', \lambda), \mu'), v - u(\mu', \lambda)\rangle \geq 0. \tag{7}$$

Using Lemma 2, either $u(\mu', \lambda) \in K_\lambda(u(\mu, \lambda))$ or there exist $w \in K_\lambda(u(\mu, \lambda))$ such that

$$|u(\mu', \lambda) - w| \leq k_2|u(\mu, \lambda) - u(\mu', \lambda)|^{\xi} \quad \text{and } |u(\mu', \lambda) - w| \geq \frac{\delta_0}{2}. \tag{8}$$

By the same argument, either $u(\mu, \lambda) \in K_\lambda(u(\mu', \lambda))$ or there exists $z \in K_\lambda(u(\mu', \lambda))$ such that

$$|u(\mu, \lambda) - z| \leq k_2|u(\mu, \lambda) - u(\mu', \lambda)|^{\xi} \quad \text{and } |u(\mu, \lambda) - z| \geq \frac{\delta_0}{2}. \tag{9}$$

Let us introduce $\bar{w}$ defined by

$$\bar{w} = \begin{cases} u(\mu', \lambda), & \text{if } u(\mu', \lambda) \in K_\lambda(u(\mu, \lambda)), \\ w, & \text{if } u(\mu', \lambda) \notin K_\lambda(u(\mu, \lambda)) \end{cases} \tag{10}$$

and $\bar{z}$ defined by

$$\bar{z} = \begin{cases} u(\mu, \lambda), & \text{if } u(\mu, \lambda) \in K_\lambda(u(\mu', \lambda)), \\ z, & \text{if } u(\mu, \lambda) \notin K_\lambda(u(\mu', \lambda)). \end{cases} \tag{11}$$

Now, by choosing $v = \bar{w}$ in (6) and $v = \bar{z}$ in (7) we obtain

$$\langle C(u(\mu, \lambda), \mu), \bar{w} - u(\mu, \lambda)\rangle \geq 0, \tag{12}$$

$$\langle C(u(\mu', \lambda), \mu'), \bar{z} - u(\mu', \lambda)\rangle \geq 0. \tag{13}$$

By strong monotonicity of $C(\cdot, \mu')$ we write

$$\begin{aligned} m|u(\mu, \lambda) - u(\mu', \lambda)|^2 \leq &-\langle C(u(\mu, \lambda), \mu'), u(\mu', \lambda) - u(\mu, \lambda)\rangle \\ &-\langle C(u(\mu', \lambda), \mu'), u(\mu, \lambda) - u(\mu', \lambda)\rangle. \end{aligned}$$

Combining (12) and (13), and the last inequality we have

$$\begin{aligned} m|u(\mu,\lambda)-u(\mu',\lambda)|^2 \le\ & \langle C(u(\mu',\lambda),\mu'),\bar{z}-u(\mu',\lambda)\rangle \\ & -\langle C(u(\mu',\lambda),\mu'),u(\mu,\lambda)-u(\mu',\lambda)\rangle \\ & +\langle C(u(\mu,\lambda),\mu),\bar{w}-u(\mu,\lambda)\rangle \\ & -\langle C(u(\mu,\lambda),\mu),u(\mu',\lambda)-u(\mu,\lambda)\rangle \\ & +\langle C(u(\mu,\lambda),\mu),u(\mu',\lambda)-u(\mu,\lambda)\rangle \\ & -\langle C(u(\mu,\lambda),\mu'),u(\mu',\lambda)-u(\mu,\lambda)\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} m|u(\mu,\lambda)-u(\mu',\lambda)|^2 \le\ & \langle C(u(\mu',\lambda),\mu'),\bar{z}-u(\mu,\lambda)\rangle \\ & +\langle C(u(\mu,\lambda),\mu),\bar{w}-u(\mu',\lambda)\rangle \\ & +\langle C(u(\mu,\lambda),\mu)-C(u(\mu,\lambda),\mu'),u(\mu',\lambda)-u(\mu,\lambda)\rangle. \end{aligned}$$

We observe that if $u(\mu,\lambda)\in K_\lambda(u(\mu',\lambda))$ then

$$\langle C(u(\mu,\lambda),\mu),\bar{z}-u(\mu,\lambda)\rangle = 0, \tag{14}$$

whereas if $u(\mu,\lambda)\notin K_\lambda(u(\mu',\lambda))$,

$$\langle C(u(\mu,\lambda),\mu),\bar{z}-u(\mu,\lambda)\rangle = \langle C(u(\mu,\lambda),\mu),z-u(\mu,\lambda)\rangle. \tag{15}$$

Analogously, if $u(\mu',\lambda)\in K_\lambda(u(\mu,\lambda))$ then

$$\langle C(u(\mu,\lambda),\mu),\bar{w}-u(\mu',\lambda)\rangle = 0 \tag{16}$$

and if $u(\mu',\lambda)\notin K_\lambda(u(\mu,\lambda))$, we have

$$\langle C(u(\mu,\lambda),\mu),\bar{w}-u(\mu',\lambda)\rangle = \langle C(u(\mu,\lambda),\mu),w-u(\mu',\lambda)\rangle. \tag{17}$$

Without loss of generality, we can suppose that w and z exist. Therefore, thanks to (h_2), (8), (9), and Remark 3 we get

$$\begin{aligned} m|u(\mu,\lambda)-u(\mu',\lambda)|^2 \le\ & b|w-u(\mu',\lambda)|^\beta + |C(u(\mu',\lambda),\mu') \\ & -C(u(\mu',\lambda),\mu)||u(\mu,\lambda)-u(\mu',\lambda)| \\ & +b|z-u(\mu,\lambda)|^\beta. \end{aligned} \tag{18}$$

From assumption (h_3) it results

$$\begin{aligned} m|u(\mu,\lambda)-u(\mu',\lambda)|^2 \le\ & 2k_2^\beta b|u(\mu,\lambda)-u(\mu',\lambda)|^{\xi\beta} \\ & +c\|\mu-\mu'\|^\gamma|u(\mu,\lambda)-u(\mu',\lambda)|. \end{aligned}$$

Accordingly (h_4), leads to

$$\begin{aligned} m|u(\mu,\lambda)-u(\mu',\lambda)|^2 \le\ & 2k_2^\beta b|u(\mu,\lambda)-u(\mu',\lambda)|^2 \\ & +c\|\mu-\mu'\|^\gamma|u(\mu,\lambda)-u(\mu',\lambda)|. \end{aligned}$$

Equivalently,

$$(m-2k_2^\beta b)|u(\mu,\lambda)-u(\mu',\lambda)|^2 \le c\|\mu-\mu'\|^\gamma|u(\mu,\lambda)-u(\mu',\lambda)|.$$

Consequently,

$$(m-2k_2^\beta b)|u(\mu,\lambda)-u(\mu',\lambda)| \le c\|\mu-\mu'\|^\gamma.$$

Let us set $c_1 = [c/(m - 2k_2^{\beta} b)]$ and $d_1 := \gamma$. Thanks to (h_4) we have that $d_1 < 1$. Therefore,

$$|u(\mu, \lambda) - u(\mu', \lambda)| \leq c_1 \|\mu - \mu'\|^{d_1}. \tag{19}$$

Now, for each μ around $\bar{\mu}$, we prove that $\lambda \mapsto u(\mu, \lambda)$ is Hölder continuous around $\bar{\lambda}$. Thus, let $\lambda, \lambda' \in \mathcal{V}(\bar{\lambda})$. Of course (2), ensures that

$$\exists u_1 \in K_{\lambda}(u(\mu, \lambda)) \text{ such that } |u_1 - u(\mu, \lambda')| \leq k_1 \|\lambda - \lambda'\|^{\xi'}, \tag{20}$$

$$\exists u_2 \in K_{\lambda'}(u(\mu, \lambda)) \text{ such that } |u_2 - u(\mu, \lambda)| \leq k_1 \|\lambda - \lambda'\|^{\xi'}. \tag{21}$$

Thanks to Lemma 2, either $u_1 \in K_{\lambda}(u(\mu, \lambda))$ or there exists $u_1' \in K_{\lambda}(u(\mu, \lambda))$ such that

$$|u_1 - u_1'| \leq k_2 |u(\mu, \lambda) - u(\mu, \lambda')|^{\xi} \quad \text{and} \quad |u_1 - u_1'| \geq \frac{\delta_0}{2}. \tag{22}$$

A similar argument allows to say that either $u_2 \in K_{\lambda'}(u(\mu, \lambda'))$ or there exists $u_2' \in K_{\lambda'}(u(\mu, \lambda'))$ such that

$$|u_2 - u_2'| \leq k_2 |u(\mu, \lambda) - u(\mu, \lambda')|^{\xi} \quad \text{and} \quad |u_2 - u_2'| \geq \frac{\delta_0}{2}. \tag{23}$$

Let us define $\bar{u_1}$ and $\bar{u_2}$ as follows

$$\bar{u_1} = \begin{matrix} u_1, & \text{if } u_1 \in K_{\lambda}(u(\mu, \lambda)), \\ u_1', & \text{if } u_1 \notin K_{\lambda}(u(\mu, \lambda)). \end{matrix} \tag{24}$$

$$\bar{u_2} = \begin{matrix} u_2, & \text{if } u_2 \in K_{\lambda'}(u(\mu, \lambda')), \\ u_2', & \text{if } u_2 \notin K_{\lambda'}(u(\mu, \lambda')). \end{matrix} \tag{25}$$

Since $u(\mu, \lambda)$ (respectively, $u(\mu, \lambda')$) is a solution to $(\text{QVI}_{\mu,\lambda})$ (respectively, $(\text{QVI}_{\mu,\lambda'})$), then we have

$$\langle C(u(\mu, \lambda), \mu), \bar{u_1} - u(\mu, \lambda) \rangle \geq 0, \tag{26}$$

$$\langle C(u(\mu, \lambda'), \mu), \bar{u_2} - u(\mu, \lambda') \rangle \geq 0. \tag{27}$$

Again, thanks to strong monotonicity of $C(\cdot, \mu)$ it results that

$$\begin{aligned} m|u(\mu, \lambda) - u(\mu, \lambda')|^2 \leq & -\langle C(u(\mu, \lambda), \mu), u(\mu, \lambda') - u(\mu, \lambda) \rangle \\ & -\langle C(u(\mu, \lambda'), \mu), u(\mu, \lambda) - u(\mu, \lambda') \rangle. \end{aligned} \tag{28}$$

Using (26)–(28) we get

$$\begin{aligned} m|u(\mu, \lambda) - u(\mu, \lambda')|^2 \leq & \langle C(u(\mu, \lambda), \mu), \bar{u_1} - u(\mu, \lambda) \rangle \\ & -\langle C(u(\mu, \lambda), \mu), u(\mu, \lambda') - u(\mu, \lambda) \rangle \\ & +\langle C(u(\mu, \lambda'), \mu), \bar{u_2} - u(\mu, \lambda') \rangle \\ & -\langle C(u(\mu, \lambda'), \mu), u(\mu, \lambda) - u(\mu, \lambda') \rangle. \end{aligned}$$

Accordingly,

$$\begin{aligned} m|u(\mu, \lambda) - u(\mu, \lambda')|^2 \leq & \langle C(u(\mu, \lambda), \mu), \bar{u_1} - u(\mu, \lambda') \rangle \\ & +\langle C(u(\mu, \lambda'), \mu), \bar{u_2} - u(\mu, \lambda) \rangle. \end{aligned}$$

Hence,

$$m|u(\mu,\lambda)-u(\mu,\lambda')|^2$$
$$\leq \langle C(u(\mu,\lambda),\mu),\bar{u_1}-u_1\rangle + \langle C(u(\mu,\lambda),\mu),u_1-u(\mu,\lambda')\rangle \quad (29)$$
$$+\langle C(u(\mu,\lambda'),\mu),\bar{u_2}-u_2\rangle + \langle C(u(\mu,\lambda'),\mu),u_2-u(\mu,\lambda')\rangle. \quad (30)$$

From (h_2), (20), and (21) it follows that

$$C(u(\mu,\lambda),\mu),u_1-u(\mu,\lambda')\rangle \leq b_0|u_1-u(\mu,\lambda')| \leq b_0k_1\|\lambda-\lambda'\|^{\xi'}, \quad (31)$$

$$\langle C(u(\mu,\lambda'),\mu),u_2-u(\mu,\lambda')\rangle \leq b_0|u_2-u(\mu,\lambda)| \leq b_0k_1\|\lambda-\lambda'\|^{\xi'}. \quad (32)$$

On the other hand, we observe that

$$\langle C(u(\mu,\lambda),\mu),\bar{u_1}-u_1\rangle = \begin{matrix} 0, & \text{if } u_1\in K_\lambda(u(\mu,\lambda)), \\ \langle C(u(\mu,\lambda),\mu),u_1'-u_1\rangle, & \text{if } u_1\notin K_\lambda(u(\mu,\lambda)), \end{matrix}$$

$$\langle C(u(\mu,\lambda'),\mu),\bar{u_2}-u_2\rangle = \begin{matrix} 0, & \text{if } u_2\in K_{\lambda'}(u(\mu,\lambda')) \\ \langle C(u(\mu,\lambda),\mu),u_2'-u_2\rangle, & \text{if } u_2\notin K_{\lambda'}(u(\mu,\lambda')). \end{matrix}$$

Suppose, without loss of generality, that u_1' and u_2' exist. Then (22), (23), and Remark 3 ensure that

$$\langle C(u(\mu,\nu_i),\mu),u_i'-u_i\rangle \leq b|u_i'-u_i|^\beta \leq bk_2^\beta|u(\mu,\lambda)-u(\mu,\lambda')|^{\xi\beta} \quad (33)$$

for all $i\in\{1,2\}$, where $\nu_1=\lambda$ and $\nu_2=\lambda'$. Using (29), (31) and (32), we conclude that

$$m|u(\mu,\lambda)-u(\mu,\lambda')|^2 \leq 2b_0k_1\|\lambda-\lambda'\|^{\xi'} + 2bk_2^\beta|u(\mu,\lambda)-u(\mu,\lambda')|^{\xi\beta}. \quad (34)$$

Now, we involve (h_4) and deduce that

$$(m-2bk_2^\beta)|u(\mu,\lambda)-u(\mu,\lambda')|^2 \leq 2b_0k_1\|\lambda-\lambda'\|^{\xi'}.$$

Therefore,

$$|u(\mu,\lambda)-u(\mu,\lambda')| \leq \left(\frac{2k_1b_0}{m-2bk_2^\beta}\right)^{1/2}\|\lambda-\lambda'\|^{\xi'/2}. \quad (35)$$

Set $c_2=[\frac{2k_1b_0}{m-2bk_2^\beta}]^{1/2}$ and $d_2=\xi'/2$.

Now, using (19) and (35) we get

$$|u(\mu,\lambda)-u(\mu',\lambda')| \leq |u(\mu,\lambda)-u(\mu',\lambda)| + |u(\mu,\lambda)-u(\mu',\lambda')|$$
$$\leq c_1\|\mu-\mu'\|^{d_1} + c_2\|\lambda-\lambda'\|^{d_2}.$$

Finally, choosing $(\mu,\lambda)=(\mu',\lambda')$ in the above expression, the uniqueness of the perturbed solution follows. We also remark that, due to condition (h_1), for each solution u to (QVI), perturbed problems $(\text{QVI}_{\mu,\lambda})$, and $(\text{QVI}_{\mu',\lambda'})$ admit unique solutions in a neighborhood of u. Thus, the proof is complete. □

Remark 4 The above result remains true even in infinite dimensional Hilbert spaces.

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